6

Inventing and Reinventing Ideas: Constructivist Teaching and Learning in Mathematics

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When the parent organization of ASCD, the National Conference on Educational Methods, published its first yearbook in 1928, the dominant view of learning expressed by education researchers was that people learn by forming connections between environmental stimuli and useful responses. This view had developed from the work of associationists like E.B. Thorndike (1922), who recommended that in mathematics, for example, students do lots of drill and practice on correct procedures and facts to strengthen correct mental bonds and habits. At the same time, associationists said, curriculums should be structured to keep related concepts well separated, so that students did not form incorrect bonds. Thorndike argued for a science of education built on experimental methods, and he suggested the need to design objective measures of students' learning in the form of valid and reliable test items.

By 1943, the behaviorists were asserting that a real science of education could only be built on direct observation. Absent from the

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research and discourse of behaviorists were "meaning," "thinking," or other such unobservable and possibly nonexistent phenomena. Though behaviorists, led by B.F. Skinner, denied the theory of "mental bonds" that associationists had put forth, their prescriptions for mathematics teaching were similar: plenty of drill and practice, with reinforcement by reward for desirable behavior (i.e., correct answers) and extinguishing or punishment for undesired behavior.

Programmed learning curriculums developed by the behaviorists, combined with the new standardized testing techniques developed by psychometricians from achievement and aptitude measures used to evaluate draftees for the U.S. Army during World War II, offered hope for a true "science" of education. Educational research promised to discover curricular materials and teaching methods that could be used by trained teachers to produce learning in students in much the way that newly developed machines were being used in factories to produce ever-increasing numbers and types of manufactured goods, and accompanying tests that could measure the exact degree of learning produced. Extending the behavioral view of learning to the study of teaching, "process-product" researchers searched for the types of teaching behavior that led to greater student achievement, under the assumption that with such a list, they could construct a prescription for effective teaching (Gage 1963, Dunkin and Biddle 1974).

Yet there existed other views of knowledge and learning during these same years, acknowledged alternatives in the scholarly community, although not dominant in the policies and practices of public schooling (Lagemann 1989, Darling-Hammond and Snyder 1992). As early as 1895, John Dewey wrote with James McLellan: "Number is not a property of objects which can be realized through the mere use of the senses or impressed upon the mind . . . . Objects (and measured things) aid the mind in its work of constructing numerical ideas" (McLellan and Dewey 1895, p. 24). In 1935, William Brownell wrote about a theory of instruction that "makes meaning, the fact that children shall see sense in what they learn, the central issue in arithmetic instruction" (p. 19). Later, based on detailed interviews with hundreds of children, Piaget and his coworkers proposed that children "make sense" in ways very different from adults, and that they learn through the process of trying to make things happen, trying to manipulate their environment (Piaget 1970).

Today, theories like these, which hold that "people are not recorders of information, but builders of knowledge structures" (Resnick and

Klopfer 1989, p. 4), have come to be grouped under the heading of "constructivism."[1]

Reemergence of Constructivism Within the Context of Reform

Over the past two decades, disappointment with public schools has been mounting; calls for reform are increasingly heard. The goal of "producing" learning in all children seems to be ever receding. Arguments have multiplied about the validity of "scientific" measures of learning, especially as applied to various nonmainstream groups, such as minorities and disadvantaged students. Schools are called on to help students learn in increasingly complex ways, because in their lives and work and thought, people do not need simply to be able to recall facts or preset procedures in response to specific stimuli. They need to be able to plan courses of action, weigh alternatives, think about problems and issues in new ways, converse with others about what they know and why, and transform and create new knowledge for themselves; they need, in short, to be able "to make sense" and "to learn."

At the same time, dissatisfaction has been growing within scholarly communities with behaviorist models of learning and objectivist views of knowledge or truth (Kuhn 1970, Lakatos 1970; Toulmin 1985). Psychologists are focusing less on the simple conditioned responses that humans share with many animals and more on *the uniquely human aspects of learning in language, art, science, mathematics, cultural groups, and societal institutions* (e.g., Resnick and Klopfer 1989). In addition, scholars are rethinking their views of knowledge, moving away from the idea that we can know something "objectively," and toward the idea that *knowledge is necessarily subjective, interpretive, and contextualized*. For these reasons, education scholars have been increasingly interested in the ideas about learning that were advanced by people like Dewey, Brownell, and Piaget, thinkers who put forth constructivist ideas. In addition, scholars are interested in more recent, "social" aspects of constructivism that portray inquiry and the growth of knowledge as occurring within communities through the processes of conversation, argumentation, justification, and "proof" (Lakatos 1976; Vygotsky 1978).

[1]The views expressed in ASCD yearbooks have, as a whole, tended to favor these alternative views. We would refer those interested to the yearbooks of 1949, 1954, 1959, 1963, and 1967, particularly.

Why "Unpack" Constructivism?

Currently, most educational scholars espouse the idea that knowledge is constructed, and much current reform rhetoric in the United States is couched in terms of "constructed knowledge" (e.g., National Research Council 1989, Rutherford and Ahlgren 1990). Although "the initial statement 'I am a constructivist' has become a kind of academic lip service" (Bauersfeld 1991, p. 3), the terms *constructivist* and *constructivism* can have many meanings. Not only do different scholars who use these terms hold differing assumptions about knowledge and how one comes to know, but these assumptions and the ways in which they might influence school teaching and learning are often not made explicit.

Those within a community of scholars are usually aware of the views and assumptions that underlie the statements and work of their colleagues within the community, as well as the views and assumptions of scholars in other communities. But those outside the scholarly community typically remain unaware. For example, educators such as principals, teachers, and curriculum designers are often presented with surface-level suggestions about how they might change toward more "constructivist" practices in their schools, without being made privy to the assumptions or theoretical frames of the various authors of these reforms, who may include researchers, policy reformers, textbook writers, or expert practitioners.

Some may protest that practicing educators are more interested in practical features than in theories; but evidence exists that, for example, teachers' enactments of suggested reforms are profoundly influenced by the theories and beliefs that they currently hold (Ball 1990, Cohen 1990, Wiemers 1990, Wilson 1990). This body of research on teachers' "reading" of reforms suggests, as does research on the reading of texts, that readers interpret texts (or reform recommendations) in light of their existing assumptions and frames. If not privy to the underlying assumptions and understandings of the author, readers may attempt to incorporate the "new information" without reexamining their existing understanding. Educators who are expected to "implement" surface features of constructivist reforms without being given time and access to consider and interpret for themselves the assumptions and ideas about learning that underlie these reforms may miss the main meaning of the reform, while adhering to the letter of the suggested procedures. Teachers, particularly, may be caught in a net of conflicting expectations, as the remnants of older reforms based on more behaviorist views remain in place at the same time constructivist-based instructional activities are urged on them (Darling-Hammond 1990, Peterson 1990). Thus teachers may come to see themselves as responsible both for students' getting the "right answers" on standardized exams and simultaneously encouraging students to explore "multiple ways of knowing" in class.

Finally, constructivist theories, like all theories of teaching and learning, pose their own dilemmas for educators (Lampert 1985, Ball in press). These dilemmas arise in specific contexts, as teachers try to help particular students learn particular things in particular classrooms and schools; thus, the dilemmas cannot be resolved in advance by the "designers" of any reform. They must be resolved again and again by practicing educators as they deal with their own particular situations. The success of all these reforms ultimately depends on the wisdom of practicing educators—their understanding of and ability to flexibly interpret constructivist ideas.

Why Explore Cases of Constructivist Mathematics Teaching?

In this chapter, we consider two examples of constructivist mathematics teaching and learning that have been created by two elementary schoolteachers working within their own communities of discourse and learning.

We have chosen examples from mathematics primarily because this is the subject area with which we are most familiar; yet we see similar questions and issues emerging in constructivist teaching in other subject areas, including literacy and science.

We have chosen to look at examples of teaching for two reasons. First, it is in the classroom interactions among teacher and students that school learning finally does or does not occur. All the planning and resources of schools, all the vital activities of administrators, curriculum specialists, supervisors, counselors, and other practicing educators in our school systems lead up to and make possible the learning that we hope will occur in the classroom through the direct mediation of the teacher. Yet, and this is our second reason, teachers are often the most excluded from the scholarly discourse around issues of teaching and learning (Carter 1992).

This absence of teachers' voices seems to reflect a dominant view of knowledge over the past fifty years—knowledge was thought to be

constructed by experts (researchers) and transmitted to practitioners (teachers), just as knowledge was thought to be constructed by experts (teachers and adults) and transmitted to novices (students). Just as some educators are challenging this transmission view of knowledge for students in our nation's classrooms, educators are also challenging it for teachers in our nation's schools (Lieberman 1992). Just as students need to think for themselves, so do teachers; and just as students need to be lifelong learners of new knowledge, so do teachers (Carnegie Forum on Education and the Economy 1986, Holmes Group 1990). Much dialogue and debate in scholarly and professional communities in education is now concerned with questions of whether and how teachers will be included in the ongoing discourse that is constructing a knowledge base for teaching and who will assume the roles of authorities for knowledge in the fields of teaching and learning (e.g., Carter 1992).[2]

In the cases in this chapter, both the perspectives and the voices of these two teachers are present and visible. We explore the assumptions about mathematics learning that these teachers bring to their mathematics teaching, as well as the assumptions of the researchers with whom they have worked. Although these two teachers had never met and were unacquainted with each others' practices, they independently created instructional practices that have both striking similarities and interesting differences.

One way of thinking about these cases is to consider some common themes, similar to the "common threads" identified by Davis, Maher, and Noddings (1990), including

> the emphasis on mathematical activity in a mathematical community. It is assumed that learners have to construct their own knowledge—individually and collectively. Each learner has a tool kit of conceptions and skills with which he or she must construct knowledge to solve problems presented by the environment. The role of the community—other learners and teacher—is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction. Any form of activity that takes place in a genuine community is likely to be complex. Initiates have to learn the language, customs, characteristic problems, and tools of the community, and there is a continual need to negotiate and renegotiate meaning. Because student communities necessarily lack the experience and authority of expert communities, teachers bear a great responsibility for guiding student activity, modeling mathematical behavior, and providing the examples and counterexamples that will turn student talk into useful communication about mathematics (p. 3).

[2]In line with recent concern for teachers' voices, both the 1990 and 1991 ASCD yearbooks include chapters authored by classroom teachers.

Another aspect of these cases is the dilemmas or tensions that emerge as these teachers work to embody constructivist theories in their practices. These cases can serve as sites for exploration for teachers who want to move toward constructivist teaching and learning in their own classrooms. They can also be used by supervisors, administrators, curriculum developers, and teacher educators to consider what kinds of resources might support teachers' attempts to teach in this way. In our ongoing research on policy and practice in more than fifty teachers' classrooms in California and Michigan, we see similar themes and dilemmas emerging as other elementary schoolteachers move toward more constructivist teaching in literacy and mathematics. However, we want readers to see these cases as "instances" of constructivist teaching, rather than "models" to be imitated. A constructivist view of knowledge implies that knowledge is continuously created and reconstructed so that there can be no template for constructivist teaching. Just as teachers' knowledge is developing and changing as teachers learn from their learners and their teaching, so too would teachers continuously recreate and transform their own teaching within their own contexts.

A third way of thinking about these cases is as texts, situated within social, cultural, and historical contexts, that may be interpreted in multiple ways by readers who also exist within such contexts. If we would practice what we preach, we cannot claim here to present *the* definitive interpretation of these excerpts, but rather simply to share some of the ideas we have about them. We expect, and indeed hope, that these cases will elicit other ideas and interpretations from other readers and thereby facilitate discourse among practitioners, policymakers, and researchers aimed at developing shared understandings and new ways to think about reform, research, teaching, and learning.

Examples of Constructivist Teaching and Learning

Deborah Ball and Annie Keith are both elementary schoolteachers who are involved in multiple communities of inquiry and discourse that include teachers, teacher educators, and researchers, as well as the students in their own classrooms. We chose these teachers because they have three important things in common. First, both teachers take social constructivist perspectives on learners' mathematical knowledge, although they have come to these views from different directions. Second,

both teachers are striving to create teaching practices that are in line with the visions of teaching in the *Standards* recently published by the National Council of Teachers of Mathematics (NCTM 1991). Third, both teachers are learners themselves, and as such they are striving continuously to renew and reform their own classroom practices.

We begin each case with a short introduction to each teacher and some of her goals in teaching, followed by excerpts from a lesson, including italicized commentary, and end with an investigation of some of the issues in constructivist teaching that seem to arise out of each excerpt.

Understanding "Sean Numbers" in Deborah Ball's Class

Deborah Ball has seventeen years of experience as an elementary schoolteacher. After teaching for eight years, she returned to school and earned a Ph.D. degree in 1988. She is currently a professor of teacher education and researcher at Michigan State University, while she continues to teach mathematics daily to a class of 3rd graders. Throughout her years of teaching, Ball has worked to improve mathematics teaching. She is one of the authors of the *NCTM Professional Standards for Teaching Mathematics* (1991). With Magdalene Lampert, Ball has had a grant from the National Science Foundation (NSF) to study her own mathematics teaching and to develop videodisc materials for teacher education.

Ball has written extensively about her experiences in trying to create and revise her teaching practice. Like Magdalene Lampert (1990), she attempts to develop a "practice that respects both the integrity of mathematics as a discipline *and* of children as mathematical thinkers" (Ball 1990, p. 3). She strives to create a classroom environment in which the norms of discourse are informed by patterns of discourse in the mathematics community as well as by the culture of the classroom. Further, she strives to shift authority for mathematical knowledge from the teacher and the "text" to the community of knowers and learners of mathematics in her classroom. She also assumes that students are "sense makers" and that, as their teacher, she needs to understand their understandings.

Ball teaches 3rd grade mathematics at Spartan Village Elementary School in East Lansing, Michigan. The school has an ethnically and linguistically diverse student body; children in the school speak twenty different languages, and many attend English-as-a-Second-Language (ESL) classes. Most of the children's parents are undergraduate or graduate students who are attending Michigan State University and live in University-subsidized student housing. The following is excerpted from a whole-class discussion of odd and even numbers (Ball 1991, in press). Ball's own comments are in italics; the names of the students are pseudonyms.[3]

> *We had been working with patterns with odd and even numbers. One day as we began class, Sean announced:*
>
> Sean:[4] I was just thinking about 6, that it's a . . . I 'm just thinking it can be an odd number, too, 'cause there could be 2, 4, 6, and two, three 2s, that'd make 6.
>
> Ball: Uh-huh . . .
>
> Sean: And two 3s, that it could be an odd and an even number. Both! Three things to make it and there could be two things to make it . . . .
>
> Ball: Other people's comments?
>
> Cassandra: I disagree with Sean when he says that 6 can be an odd number. I think 6 can't be an odd number because, look . . . [she goes to the board and points to the number line there, starting with zero] even, odd, even, odd, even, odd, even. How can it be an odd number because . . . zero's not an odd number [appealing to an implicit definition of even numbers as 'every other number'] . . . .
>
> Ball: What's the definition—Sean?—what's our working . . . definition of an even number? . . .
>
> *At this point I thought that Sean was just confused about the definition for even numbers. I thought that if we just reviewed the definition, he would see that 6 fit the definition and was therefore even . . . .* (There are several minutes spent recalling and discussing the working definition. Agreement is reached).
>
> Jeannie: If you have a number that you can split up evenly without having to split one in half, then it's an even number.
>
> Ball: Can you do that with 6, Sean? Can you split 6 in half without having to use halves?
>
> Sean: Yeah.
>
> Ball: So then it would fit our working definition, then it would be even. Okay?
>
> Sean: [pause] And it could be odd. Three 2s could make it . . . . It fits the definition for odd, too.
>
> Ball: What is the definition for odd? Maybe we need to talk about that?

[3]This selection is excerpted from two sources in which Ball discusses this lesson (Ball 1991; Ball in press).

[4]Although in this chapter we have used Deborah Ball's and Annie Keith's real names, according to their wishes, all student names are pseudonyms.

> *We discussed a definition for odd numbers [and] we agreed that odd numbers were numbers that you could not split up fairly into two groups. But this still did not satisfy Sean . . . .*
>
> Sean: You could split 6 fairly, *and* you can split 6 not fairly . . . . Like, say there's 2 of you, and you had 6 cookies, and you didn't want to split them in half . . . you wanted to split them by 2s. Each person would get 2 and there would be 2 left . . . .
>
> Ball: So, are you saying all numbers are odd, then?
>
> Sean: No, I'm not saying all numbers are odd, but . . .
>
> Ball: Which numbers are *not* odd then?
>
> Sean: Um . . . 2, 4, 6, . . . 6 can be odd or even . . . 8 . . .
>
> Students: No!
>
> Temba: Prove it to us that it can be odd. Prove it to us.
>
> Sean: Okay. [He goes to the board.] Well, see, there's two [he draws] number 2 over here, put that there. Put this here. There's 2, 2, and 2, and that would make 6.
>
> **OO | OO | OO**
>
> Temba: I know, which is even!
>
> Mei: I think I know what he's saying . . . . I think what he's saying is that you have 3 groups of 2. And 3 is an odd number, so 6 can be an odd number and an even number.
>
> Ball: Do other people agree with that? Is that what you're saying, Sean?
>
> Sean: Yeah.
>
> Ball: Okay, do other people agree with him? [pause] Mei, you disagree with that?
>
> Mei: Yeah, I disagree with that because it's not according to like . . . how many groups it is. Let's say I have [pauses] Let's see. If you call 6 an odd number, why don't [pause] let's see [pause] let's see—10. One, two . . . [draws circles on the board] and here are 10 circles. And then you would split them, let's say I wanted to split them by 2s . . . . 1, 2, 3, 4, 5 [she draws].
>
> **OO | OO | OO | OO | OO**
>
> Then why do you not call 10 an odd number and an even number, or why don't you call other numbers an odd number and an even number?
>
> Sean: I didn't think of it that way. Thank you for bringing it up, so—I say it's—10 can be an odd and an even.
>
> Mei: [with some agitation] What about other numbers? Like, if you keep on going on like that and you say that other numbers are odd and even maybe we'll end it up with all numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because

> if all numbers were odd and even, we wouldn't be even having this discussion!

In this excerpt, Deborah Ball deals with two of the dilemmas, or issues, common to constructivist teaching. First, what is the teacher's role in constructivist learning? Second, how can the teacher honor both student-constructed knowledge and traditionally accepted knowledge?

What is the teacher's role in constructivist learning?

Constructivist theory holds that learning involves students' constructing their own knowledge. Yet students cannot be expected to construct centuries' worth of knowledge all on their own. One of the "common threads" in constructivism identified by Davis and colleagues (1990) concerns this redefinition of the teacher's role, away from *directing* all classroom discourse and *telling* students correct procedures and right answers, toward "*guiding* student activity, *modeling* mathematical behavior, and *providing* the examples and counterexamples that will turn student talk into useful communication about mathematics" (p. 3, emphasis added).

This episode from Deborah Ball's teaching reveals one way of handling this new role. The teacher was quite active in the class discussion: she clarified students' remarks, posed challenging questions, and thought hard about where she wanted the discussion to go. At the same time, the discussion was in large part shaped by the students' concerns. Sean made the original conjecture that "some numbers can be odd or even." Other students argued with Sean's conjecture, expanded it, demanded proof of it, and discussed its significance for definitions of odd and even numbers in mathematics. Although Ball was a major participant and, at times, moderator of the discussion, she maintained her posture that authority for mathematical knowledge should reside with the community of learners in her classroom. The entire classroom community, through mathematical argument, justification, and sense-making, wrestled with just how and whether Sean's conjecture would be accepted. Ball's rationale was that

> *in traditional classrooms, answers are right most often because the teacher says so . . . . I am searching for ways to construct classroom discourse such that the students learn to rely on themselves and on mathematical argument for resolving mathematical sense.*

How can the teacher honor both student-constructed knowledge and traditionally accepted knowledge?

Constructivist ideas about teaching emphasize the importance of listening to and valuing students' perceptions, even when their understanding differs from conventional knowledge. Listening and valuing are part of the vital "support" that Davis and colleagues suggest is necessary to encourage students to construct their knowledge, to engage in the hard and risky task of openly wondering, conjecturing, testing, and arguing about mathematics or any other subject. Such listening and valuing also reflect the constructivist, epistemological stance that knowledge, even "official" knowledge, is not fixed and static, but ever changing and growing. Yet students also need to understand the conventional knowledge that is currently accepted in their society, and teachers are responsible for helping them gain this understanding.

Sean's suggestion that "some numbers can be odd and even" because they contain an odd number of groups of two caused Deborah Ball to struggle hard with the dilemma of how to respect Sean's understanding, yet avoid confusing him and his classmates. She wrote:

> *On the one hand, Sean was wrong. Even and odd are defined to be non-overlapping. . . . He was . . . paying attention to something that was irrelevant to the conventional definition for even and odd numbers . . . . On the other hand . . . Sean noticed that some even numbers have an odd number of groups of two. Hence, they were, to him, special. . . . I wrote in my journal: "I'm wondering if I should introduce to the class the idea that Sean has identified (discovered) a new category of numbers—those that have the property he has noted. We could name them after him. Or maybe this is silly—will just confuse them since it's nonstandard knowledge. . . . "*
>
> *In the end, I decided not to label his claim wrong, and, instead, to legitimize Sean's idea of a number that can be "both even and odd." I pointed out that Sean had invented another kind of number that we hadn't known before and suggested that we call them "Sean numbers.". . . And, over the course of the next few days, some children explored patterns with Sean numbers, just as others were investigating patterns with even and odd numbers.*

Ball's decision to trust her students' ability to understand and discriminate worked out well. She comments:

> *When I gave a quiz on even and odd numbers . . . the results were reassuring. Everyone was able to give a sound definition of odd numbers, and to correctly identify and justify even and odd numbers. And, interestingly, in a problem that involved placing some numbers into a string picture (Venn diagram), no one placed 90 (a Sean number) into the intersection between even and odd numbers. If they were confused about these classifications of number, the quizzes did not reveal it.*

Ball also learned from this episode as she participated in the classroom discussion and came to understand Sean's idea. She learned a lot about how Sean and his classmates were thinking about odd and even numbers, and she also learned something about mathematics from Sean. Although Ball had never heard of "Sean numbers"—numbers composed of an odd number of groups of two—when Sean "discovered" them in class, she subsequently found out that Greek mathematicians had discovered this kind of number and worked with it centuries ago. Janine Remillard, a graduate student and colleague of Ball's, called this to her attention. Remillard found in D.E. Smith's *History of Mathematics*, Vol. II, the following:

> Euclid [studied] "even-times-even numbers," "even-times-odd numbers," and "odd-times-odd numbers." His definitions of the first two differ from those given by Nicomachus (*c.* 100) and other writers. . . . How far back these ideas go in Greek arithmetic is unknown, for they were doubtless transmitted orally long before they were committed to writing (p. 18).

Harvey Davis, of the mathematics department at Michigan State University, called to our attention that both Plato and the neo-Pythagoreans had also worked with "Sean-type" numbers—those produced by multiplying two by an odd number, resulting in "an odd number of groups of two."

Working Together on Problems in Annie Keith's Class

The second teacher, Annie Keith, had just completed her first year of teaching when she began participating six years ago in the development of Cognitively Guided Instruction (CGI)—a research-based approach to elementary mathematics learning. Keith began by participating in a month-long workshop in 1986 and became more involved in the project with each passing year. For the past two years, she has served as a mentor teacher on the CGI Project, working with researchers to "extend the principles of CGI to the primary mathematics curriculum" (Carpenter, Fennema, and Franke 1992). The full story of Annie Keith's learning and how she came to create her current mathematics practice are explored elsewhere (Peterson 1992).

The major thesis of CGI is that children enter school with a great deal of informal, intuitive knowledge of mathematics that can serve as the basis for developing much of the formal mathematics of the primary school curriculum. Although each teacher creates her own unique practice, CGI classrooms are typically characterized by a focus on problem solving, particularly the solving of word problems; students' sharing of their diverse strategies for solving the problems; and teachers' and students' listening hard to students' solutions and ideas for solving problems.[5]

Drawing on her experience in language arts with creating a classroom "community of readers and writers," Annie Keith attempts in her mathematics teaching to help her students see themselves as a community of mathematicians. At the beginning of the year, the class jointly defined the following qualities of mathematicians:

> Mathematicians listen to each other. Mathematicians never say "can't." They will always do their best and try their hardest. Mathematicians help each other. Mathematicians can solve a problem in many ways. Mathematicians use different kinds of math tools.

These qualities were not derived from any knowledge of specific or actual communities of mathematicians. Rather, they represent Keith's and her students' ideal of how *they* want to function as a community investigating mathematical ideas.

Keith encourages her students, as mathematicians, to choose and create problems and mathematical tasks that interest and challenge them and to justify their mathematical thinking to themselves and within their community. Like Deborah Ball, Annie Keith wants authority for knowing and learning to rest with the students and the community of learners rather than with her as the teacher.

Keith teaches 1st grade at Muir School in Madison, Wisconsin. For the past twenty years or so, Muir School has served a neighborhood population of white, middle-class families, as well as an additional population of students from a nearby low-rent housing area. Over the years, the latter population has changed to include a substantial number of Indo-Chinese immigrants, as well as Hispanics and African Americans. Currently, minority students make up about 30 percent of the school population, and an approximately equal number of children receive free or reduced-price lunch.

Each day, mathematics class starts with students sitting on the rug for a meeting or whole-class conversation. Then students go to math centers to work in small groups on different mathematics tasks. Students choose the center they want to work in for the day. The following selection is excerpted from field notes of a session near the end of the school year in the "Discussion" Center. Annie Keith was meeting with a group of four students involved in solving word problems that had

[5]For further details on this NSF-sponsored teacher enhancement and research project, see Carpenter, Fennema, Peterson, Chiang, and Loef 1989; Peterson, Fennema, and Carpenter, 1991.

been written by members of the class. Keith's comments from a follow-up interview are in italics.[6]

Annie Keith read the first problem, written by Susan: "I found 16 icicles. I found 80 more. How many do I have now?" and then asked the students to write a number sentence that showed what they're thinking about.

[I am] linking story problems and number sentences, [so that] when my kids see number sentences they're not thrown by them. . . . If we have a story problem, the kids can put it in number sentences, and they're very comfortable with the symbols.

T.J. wrote: **80 + 10 → 90 + 6 = 96**[7]

Keith looked at his solution and then asked T.J., "How can you challenge yourself?" T.J. decided to make the first number in the problem larger, changing it to: "I found 1,000,293 icicles. I found 80 more. How many do I have now?" then proceeded to work on this new, more challenging problem.

Jafari made 16 tally marks. Keith led him to count by tens to 80 and then count on, using his 16 tally marks. Jafari got 96 and then wrote: **16 + 80 = 96**

Heather had only written **16 + 80 = 96** on her paper. Keith said that she had heard Heather do some counting, and suggested that she find some way to show "where you started counting."

I'm really pushing them to write down on their papers how they've solved it—whether it's in words or whether it's with a number sentence that shows they're counting on or if they are putting numbers together.

Keith asked Peter how he had done the problem, and he indicated that he had first "known" that 80 + 10 = 90, and then figured 90 + 6 would be 96, "since 90 didn't have any other number on it."

Keith then called on Jafari to tell the kids how he did the problem. He had written on his paper:

16 + 80 = 96

10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 16 = 90

He counted out loud: "10-20-30-40-50-60-70-80-90-91-92-93-94-95-96."

[6]These data were collected by Peterson as part of her work as an external evaluator for a current NSF-funded project, "A Longitudinal Analysis of Cognitively Guided Instruction and the Primary School," E. Fennema and T. Carpenter, principal investigators.
[7]The arrow notation was invented by elementary students in a constructivist mathematics classroom in South Africa (Oliver, Murray, and Human 1991) and shared with Elizbeth Fennema and Tom Carpenter, who shared the idea with Annie Keith, who shared it with her students.

Keith chose a problem she had written herself for the next problem: "Steve had 14 snowballs. How many more snowballs will he need to make so he has 26 snowballs altogether?"

At this point, Jafari appeared not to be listening or participating. Keith turned to him and said, "May I go on? Then I need to see you're ready; can you sit down please? Jafari, mathematicians work together, okay?" She reread the problem.

Jafari went to get an abacus and brought it back to the table, sat down, and began to use it to solve the problem. He counted out 80 and then seemed to lose interest again. Keith suggested he might draw a picture to help with the problem.

Meanwhile, T.J. had written on his paper: **14 + 10 → 24 + 2 = 12**

Jafari seemed not to be interested in this task. He got up from the table and wandered around the room. Keith got up and went over to talk privately with Jafari. Jafari returned to the table with her, and he sat down to work again.

Sometimes Jafari will come to things hesitantly, where he thinks he can't do it, and he got really upset with this one. [I was] just saying, "I know you can do this stuff. You just need to decide. . . . Do you want to give it your best shot and work with us in this group, or do you want to join a group at another Center?"

The other three students were finished and started to share their solution strategies at Keith's urging. Meanwhile, still working on the word problem, Jafari had made 14 tally marks on his paper and put a number by each one in order from 1 to 14. Keith suggested that Jafari listen to Heather's explanation, that he might hear "the missing piece" to his solution; but Jafari did not seem to heed her suggestion, and he continued to work on his own solution.

Watching Jafari, Peter noted suddenly, "I think it clicked." Keith asked Jafari, "Are you ready to talk to us yet, or do you want us to come back?" and Jafari said, concentrating, "Come back." Jafari now had 15 tallies on his paper.

T.J. began explaining his solution strategy. As he began to explain, he erased, saying he "forgot something." He had written: **14 + 10 = 24 + 2 = 12**

He erased this and wrote, **14 + 10 → 24 + 2 → 26**

2 + 10 = 12

T.J. continued, "Fourteen plus ten is twenty-four plus two is twenty-six. Two plus ten equals twelve." Heather and Peter listened as T.J. recounted his solution strategy.

Meanwhile Jafari now had 22 tally marks. Keith asked, "How high do you need to go up to here?" Peter replied, "26." Jafari added more tally marks. Heather suggested that Jafari needed four more tally marks. Keith said to Jafari, "Keep going. You're almost there."

When Jafari had finished, Keith asked him, "How many more [snowballs] does he need to make?" Jafari replied, "Twelve." She asked him, "How do you know? How would you prove it to us?" Jafari separated off the original 14 tallies, and counted the remaining tallies needed to make 26, "one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve." Keith replied, "Very nice job. You did not give up . . . . Good for you. It was a very hard job, and yet you guys had it. Let's see if we can do one more, okay?"

Showing great eagerness and excitement, Jafari called out, "Do mine!" (meaning do the word problem he had written). Readily agreeing, Keith read Jafari's problem aloud: "I had one hundred snowballs. My mom gave me eight. How many do I have?"

Jafari immediately answered, "108." Keith responded that he should show how he got that. Jafari wrote in his notebook: **100 + 8 = 108**.

After the other kids had a chance to work out the problem, Keith said, "Okay, Jafari, start it off . . . " Jafari said, "One hundred plus eight equals one hundred and eight." He had written ten zeroes in his notebook, each representing a ten, and then the numbers from one to ten as follows:

0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8

Jafari counted aloud the zeroes by ten: "Ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, one hundred." Then he continued counting aloud the ones from 100 to 108: "one hundred one, one hundred two, one hundred three, one hundred four, one hundred five, one hundred six, one hundred seven, one hundred eight."

How can teachers involve diverse students in community problem solving?

In this episode, Annie Keith faced an issue common to all types of teaching: student diversity. Davis and colleagues (1990) note that "each learner has a tool kit of conceptions and skills," but each learner comes to school with different tools, depending on their personalities, cultures, and prior experiences. Annie Keith, like many constructivist teachers, placed great emphasis on children talking with and learning from each other:

Talking is a real priority in this room—getting kids to talk back and forth to each other [and] really think about what people are saying.

So Annie Keith needed to find ways to involve all four students in problem solving and enable them to confidently share their solutions, despite a wide diversity in their mathematical abilities and interests. At one extreme was T.J., who quickly solved the first problem symbolically by writing a number sentence, and at the other was Jafari, who worked more slowly, directly representing each of the quantities in the word

problem with tally marks.

Giving students options is an important way in which Annie Keith involves students in mathematical problem solving in this episode. When T.J. finished quickly, she suggested he "challenge himself" by making the problem more difficult. T.J. did so by making the numbers bigger. Meanwhile the other three students had time to complete their problems, free to use whatever strategy and math tools they wanted as long as they could articulate their thinking and justify their answers.

But letting students make choices also brings its own dilemmas, for in the middle of the problem-solving session, Jafari, a student who had transferred into the class just a few weeks earlier, chose to leave the table and involve himself in something other than mathematical problem solving. Keith faced a dilemma: Should she tell Jafari to return to the table and complete the problem-solving session or should she attempt to work within the norms that had been established for the mathematical community within her classroom?

Keith chose to work within the norms of her classroom community and to make use of her ongoing, intimate knowledge of Jafari's personal background and experiences and his developing mathematical understanding. She gave Jafari the choice of whether to return to the group, where he would be expected to participate and do his best thinking, or to choose to join another mathematics group, and she gave him this choice privately.

What I know about Jafari is that he's the kind of kid that if I confront him in a group, he might have to . . . come off macho. . . . One on one, it's very different, because he can walk back to the class without having lost face with anybody.

A short time after he returned to the group, Jafari asked if the group could work the word problem that he had written. When Keith and the group readily agreed, Jafari beamed and proceeded to solve the problem using his mathematical tools.

A second short excerpt points up a different issue that arises when students share ideas that may be challenged by others in the class.[8]

The next day, during whole-class discussion, Peter began to tell his classmates about "touchpoints" on numbers, a method of calculating

[8]This excerpt comes from Peterson's transcription and analysis of a videotape of Annie Keith's classroom taken by Susan Baker, a CGI project staff person, on the day following the "Jafari" excerpt. Keith's remarks in italics are, as before, from the follow-up interview with Peterson. The videotape constitutes data collected by CGI researchers for their current NSF-funded project. Our analysis here is in no way intended to substitute for or supplant their own analyses of these data.

the sum of written digits by counting imaginary dots or "points" on the digits, which Peter had learned from his sister.

At one point in the discussion, Peter's interested and curious 1st grade peers began peppering him with questions.

He felt a little pushed into a corner, I think. . . And then he just turned around and he just started crying.

Keith responded by putting her arm around Peter and reminding him that students in this class "ask hard questions" because "they really want to know things."

I just wanted him to realize that they weren't attacking him. They're just really curious and trying to figure out this whole thing. And Peter tends to be one who's very, very sensitive about things.

Then she gave him the chance to leave the discussion, get a drink of water in the hall, and return to the discussion when he felt comfortable.

Peter left the room, got a drink, and returned within thirty seconds. He rejoined the discussion and returned to the board, where two other students had taken over leading the discussion on touchpoints. At the end of class, Peter volunteered to find out more information about touchpoints and bring the information back to the class. Having been supported in the risk of sharing his ideas, Peter had voluntarily rejoined the community of mathematicians in the classroom.

How can teachers help students handle the risks of publicly sharing and debating ideas?

As Davis and colleagues (1990) suggest, constructivist teaching and learning involves students and teachers in "complex" discourse, communal attempts to "negotiate and renegotiate meaning" through public discussion and debate of their conjectures, ideas, methods, solutions, and questions. Like most constructivist teachers, Annie Keith strives to find ways to help students feel safe in presenting and discussing their ideas; yet she also strives to have students think and work like mathematicians—who ask hard questions, publicly wrestle with ideas, and are called on to justify their thinking.

In the preceding excerpt, Keith had to deal with the tensions produced by these goals, which might be seen as opposing. She modeled for the class her attitude that although students may have different ideas and come from diverse backgrounds, these differences are valued, and everybody can learn from each other. She showed respect for Peter's idea, even though the "touchpoint" method of addition depends much more on rote learning than the methods she personally might espouse. Yet she also helped Peter understand that part of learning is questioning and clarifying one's own ideas and those of others—that asking "Why?"

and "How do you know?" and "What do you mean by that?" are important parts of the classroom discourse. Finally, Keith offered Peter the dignity of recovering from his upset in private and trusted him with the decision of when to rejoin the group.

I think that's something that's really important, that they should know where their frustration point is. When [the students] feel that frustration point, [they] need to back off and come back at it again. . . . Kids will walk out of here and get a drink and come back and work. . . . I think that's really good to learn that as a kid, so as an adult you know where your point is.

Keith's concern for Peter's feelings, respect for his dignity, and trust in his judgment seemed a part of what helped Peter maintain his self-confidence and enthusiasm in the face of his critical-sounding peers and gave him the courage to return to the fray.

Inventing the Knowledge Needed for Teaching

We have used these two cases to investigate several issues faced by many teachers who are trying to teach in more constructivist ways. Deborah Ball and Annie Keith show some commonalities that may illuminate some general characteristics of successful constructivist teaching:

- Both teachers see themselves as learners—learning from their students, their colleagues, and their own investigations of mathematics. They both assert that they have changed and learned throughout the course of their teaching experiences, continuously creating and reinventing their practices as teachers.
- Both believe it is essential to listen to and respect students' ideas, yet also value students' coming to understand the mathematical constructions of the wider disciplinary community.
- Both want students to develop their own strategies for "sense-making," rather than depending on the authority of the teacher or text to determine what is the "right" answer.
- Both strive to involve students in a classroom community where they will learn to share, debate, construct, modify, and develop important mathematical ideas and ways of problem-solving.

Yet the unique flavor of these teachers also comes through clearly in these excerpts. Each teacher is an individual, not a carbon copy of some ideal model of a "constructivist teacher"; and each solves the specific dilemmas she encounters in her ongoing practice in her own ways. Indeed, a generalized model or prescription for constructivist education would be an oxymoron. A prescription implies a generalized, decontextualized list that would be good for all times and all situations, a set of procedures or solutions that some outside person could construct and then transmit or transfer to practicing educators. But just as students are continuously constructing new knowledge that is contextualized within a community of learners and within specific personal situations, so are teachers. Both Keith and Ball serve as examples of growth and change in their own knowledge, understanding, and teaching within their own learning and teaching contexts.

But these two cases raise a new puzzle and tension for practicing educators, researchers, and reformers to address, for they suggest new ways of thinking about the construction of a knowledge base for practice. In decades past, researchers, policymakers, and practitioners have worked within a model of knowledge in which researchers and policymakers construct knowledge and then "disseminate" or transmit this knowledge to administrators and teachers who are supposed to "implement" it in their schools and classrooms. Within a constructivist model, teachers, students, administrators, policymakers, and other educators would all be involved in "learning" and would participate with researchers in the ongoing construction of a knowledge base for practice (e.g., Cohen and Barnes in press). How might this be brought about?

Again, consider the cases of the two teachers we have discussed. Both Ball and Keith participate as active members of several learning or discourse communities (see Keith 1992). Ball has been an elementary schoolteacher for many years and is a major participant in the National Council of Teachers of Mathematics. Keith belongs to professional associations in reading and mathematics; she also participates in the community defined by the teachers in her school and, more specifically, by the 1st grade teachers on her primary team with whom she has weekly meetings to plan and construct curriculum.

Both teachers are also members of a community of educational scholars and researchers. In her research and communication with colleagues at Michigan State University, Ball participates in a rich discourse about the teaching and learning of mathematics. Keith also participates in a community of researchers and university professors centered around CGI. She often spends her days off at the CGI offices at the university, talking about her teaching with the researchers and

graduate students involved with CGI. She also helps shape this project through her participation as a mentor teacher. Through these personal contacts, both teachers have access to the current thinking, knowledge, and understandings of scholars to which most teachers would not have access, except through reading research articles or hearing a scholar give an invited address at a national conference. The benefits of these interactions are certainly not one sided, since through conversations with Ball or Keith, other scholars also have access to the knowledge, thinking, and understanding of an elementary schoolteacher, fresh from the challenges of learning and teaching a new mathematics in new ways to a diverse group of wriggling, laughing, boisterous young learners.

A third important learning community for both Ball and Keith is that of their students. We have already mentioned how Ball sees herself as constantly learning from her students, for example, from Sean about the unique characteristics of "Sean numbers." Keith describes herself as having hated mathematics to the point of being "math phobic" throughout her own schooling. In her preservice teacher education at the University of Wisconsin-Madison, she took two courses on mathematics for elementary schoolteachers; but when she began teaching seven years ago, she still did not feel comfortable with mathematics. The turning point was when she became involved in the CGI project after her first year of teaching. Keith credits her 1st grade students for much of her growth in understanding, confidence, and interest in mathematics over the past six years:

> *What have I learned? I've learned how much fun math really is, and how exciting it is. I think I probably learned even this whole idea of place value with understanding through watching these kids. You know, really getting at their thinking and understanding. . . . I just find them so incredible.*

The experiences of these teachers suggest that one way practicing educators can construct a knowledge base for constructivist learning and teaching is through personally participating in diverse communities of researchers, teachers, and learners. But we do not suggest such participation is the only way. Indeed, to prescribe this as *the way* would be antithetical to constructivist views. The challenge is for scholars, administrators, teachers, and learners to work together to invent and reinvent ways in which they can construct the knowledge base needed for learning and teaching in the next fifty years.

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